On two families of near-best spline quasi-interpolants on non-uniform partitions of the real line

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January 2006

Abstract

The univariate spline quasi-interpolants (abbr. QIs) studied in this paper are approximation operators using B-spline expansions with coefficients which are linear combinations of discrete or weighted mean values of the function to be approximated. When working with nonuniform partitions, the main challenge is to find QIs which have both good approximation orders and uniform norms which are bounded independently of the given partition. Near-best QIs are obtained by minimizing an upper bound of the infinity norm of QIs depending on a certain number of free parameters, thus reducing this norm. This paper is devoted to the study of two families of near-best QIs of approximation order 3.

Keywords : spline approximation, spline quasi-interpolants.
AMS classification : 41A15, 41A35, 65D07.

1 Introduction

A spline quasi-interpolant (abbr. QI) of f has the general form

$$Qf = \sum_{\alpha \in A} \lambda_\alpha(f) B_\alpha$$

where $\{B_\alpha, \alpha \in A\}$ is a family of B-splines forming a partition of unity and $\{\lambda_\alpha, \alpha \in A\}$ is a family of linear functionals which are local in the sense that they only use values of f in some neighbourhood of Σ_α =supp(B_α). The main interest of QIs is that they provide good approximants of functions without solving any linear system of equations. In this paper, we want to study the following types of QIs:

Discrete Quasi-Interpolants (abbr. dQIs) : the linear functionals are *linear combinations of values* of f at some points in a neighbourhood of Σ_α (see e.g. [1]-[3], [6]-[9],[11],[13][14][23][24]).

Integral Quasi-Interpolants (abbr. iQIs) : the linear functionals are *linear combinations of weighted mean values* of f in some neighbourhood of Σ_α (see e.g.[2]-[5],[14],[23]-[26]).

More specifically, we study QIs that we call *Near-Best Quasi-Interpolants* (abbr. NB QIs) which are defined as follows:

1) Near-Best dQIs: assume that $\lambda_\alpha(f) = \sum_{\beta \in F_\alpha} a_\alpha(\beta) f(x_\beta)$ where the finite set of points $\{x_\beta, \beta \in F_\alpha\}$ lies in some neighbourhood of Σ_α. Then it is clear that, for $\|f\|_\infty \leq 1$ and $\alpha \in A$, $|\lambda_\alpha(f)| \leq \|a_\alpha\|_1$, where a_α is the vector with components $a_\alpha(\beta)$, from which we deduce immediately

$$\|Q\|_\infty \leq \sum_{\alpha \in A} |\lambda_\alpha(f)| B_\alpha \leq \max_{\alpha \in A} |\lambda_\alpha(f)| \leq \max_{\alpha \in A} \|a_\alpha\|_1 = \nu_1(Q).$$

Now, assuming that $n = \mathrm{card}(F_\alpha)$ for all α, we can try to find $a^*_\alpha \in \mathbb{R}^n$ solution of the minimization problem

$$\|a^*_\alpha\|_1 = \min\{\|a_\alpha\|_1;\ a_\alpha \in \mathbb{R}^n,\ V_\alpha a_\alpha = b_\alpha\}$$

where the linear constraints express that Q is exact on some subspace of polynomials. Thus, we finally obtain

$$\|Q\|_\infty \leq \nu^*_1(Q) = \max_{\alpha \in A} \|a^*_\alpha\|_1.$$

2) Near-Best iQIs: assume that $\lambda_\alpha(f) = \sum_{\beta \in F_\alpha} a_\alpha(\beta) \int_{\Sigma_\beta} M_\beta(t) f(t) dt$, where the B-splines M_β are normalized by $\int M_\beta = 1$. Note that the B-spline M_β can be different from B_α. Once again, for $\|f\|_\infty \leq 1$, we have

$$|\lambda_\alpha(f)| \leq \sum_{\beta \in F_\alpha} |a_\alpha(\beta)| |\int_{\Sigma_\beta} M_\beta(t) f(t) dt| \leq \sum_{\beta \in F_\alpha} |a_\alpha(\beta)| = \|a_\alpha\|_1$$

whence, as we obtained above for dQIs,

$$\|Q\|_\infty \leq \max_{\alpha \in A} \|a_\alpha\|_1 = \nu^*_1(Q).$$

As emphasized by de Boor (see e.g. [4], chapter XII), a QI defined on non uniform partitions has to be *uniformly bounded independently of the partition* in order to be interesting for applications. Therefore, the aim of this paper is to define some families of discrete and integral QIs satisfying this property and having the *smallest possible norm*. As in general it is difficult to minimize the true norm of the operator, *we have chosen to solve the minimization problems* defined above.

The paper extends some results of [1] [11], and is organized as follows. We first recall some "classical" QIs of various types and we verify that they are uniformly bounded. Then we define and study several families of discrete and

integral QIs, depending on a finite number of parameters, for which we can find $\nu_1^*(Q)$. We show that this problem has always a solution (in general non unique). Of particular interest are the results of theorems 1,4,6 and 8 where we show that some families of dQIs and iQIs are uniformly bounded independently of the partition. By imposing more constraints on the non-uniform partitions, we can also prove that some families of QIs are near-best (theorems 5 and 9). (A parallel study of spline QIs is done in [2] for uniform partitions of the real line and in [3] for a uniform triangulation of the plane). In all cases, the QIs that we study are only exact on $\mathbb{P}_2$, i.e. their approximation order is 3. It seems surprisingly difficult to construct QIs which are both uniformly bounded independently of the partition and exact on $\mathbb{P}_d$ for $d \geq 3$. In the last section 11, we consider an example of QI which is exact on $\mathbb{P}_3$ (i.e. of approximation order 4) and uniformly bounded: however the bound depends on the *maximal mesh ratio* of the partition. Such operators seem also useful for applications and it would be interesting to study near-best operators of this type, thus allowing a reduction of the upper bound of the norm.

2 Notations

We shall use classical B-splines of degree m on a bounded interval $I = [a, b]$ or on $I = \mathbb{R}$. For the sake of simplicity, in the case $I = \mathbb{R}$, we take a strictly increasing sequence of knots $T = \{t_i, i \in \mathbb{Z}\}$ satisfying $|t_i| \to +\infty$ as $|i| \to +\infty$. In the case $I = [a, b]$, we take the usual sequence T of knots defined by (see e.g. [4],[9]) :

$$t_{-m} = \cdots = t_0 = a < t_1 < t_2 < \cdots < t_{n-1} < b = t_n = \cdots = t_{n+m}.$$

For $J = \{0, \ldots, n+m-1\}$, the family of B-splines $\{B_j, j \in J\}$, with support $\Sigma_j = [t_{j-m}, t_{j+1}]$ is a basis of the space $S_m(I, T)$ of splines of degree m on the interval I endowed with the partition T. These B-splines form a partition of unity, i.e. $\sum_{j \in J} B_j = 1$. We set $h_i = t_i - t_{i-1}$ for all indices i.
Let $\mathbb{N}_m = \{1, \ldots, m\}$ and $T_j = \{t_{j-r+1}, r \in \mathbb{N}_m\}$: we recall that the *elementary symmetric functions* $\sigma_l(T)$ of the m variables in T_j are defined by $\sigma_0(T_j) = 0$ and for $1 \leq l \leq m$, by

$$\sigma_l(T_j) = \sum_{1 \leq r_1 < r_2 < \ldots < r_l \leq m} t_{j+1-r_1} t_{j+1-r_2} \ldots t_{j+1-r_l}.$$

Denoting $C_m^l = \frac{m!}{l!(m-l)!}$ the binomial coefficients, then, for $0 \leq l \leq m$, the monomials $e_l(x) = x^l$ can be written $e_l = \sum_{i \in J} \theta_i^{(l)} B_i$, with $\theta_i^{(l)} = \sigma_l(T_i)/C_m^l$. This is a direct consequence of Marsden's identity ([4], chapter IX). In particular, the Greville points have abscissas $\theta_i = \theta_i^{(1)}$.
Similarly, we define the *extended symmetric functions* $\bar{\sigma}_l(T_j)$ by $\bar{\sigma}_0(T_j) = 1$ and, for $1 \leq l \leq m$,

$$\bar{\sigma}_l(T_j) = \sum_{1 \leq r_1 \leq r_2 \leq \ldots \leq r_l \leq m} t_{j+1-r_1} t_{j+1-r_2} \ldots t_{j+1-r_l}.$$

Then, the moments of the B-spline M_j of degree $m-2$, with $\mathrm{supp}(M_j) = [t_{j-m+1}, t_j]$ and normalized by $\int M_j = 1$, are given by (see e.g. [17]):

$$\mu_j^{(l)} := \int x^l M_j(x)dx = (C_{m+l-1}^l)^{-1}\bar{\sigma}_l(T_j).$$

3 Uniformly bounded discrete quasi-interpolants exact on $\mathbb{P}_2$

It is possible to derive *discrete* QIs from the de Boor-Fix QIs [6] by replacing the values of derivatives $D^l f(\theta_i)/l!$ by divided differences at the points θ_i lying in Σ_i. Doing this, we loose the property of projection on $S_m(I,T)$. However, by choosing conveniently the divided differences, we can obtain some families of dQIs which are uniformly bounded and exact on specific subspaces of polynomials.
Let us construct for example a family of dQIs of degree m which are *exact on* $\mathbb{P}_2$. We start from the de Boor-Fix functionals truncated at order 2:

$$\lambda_j(f) = \frac{1}{m!}\sum_{l=0}^{2}(-1)^{m-l}D^{m-l}\psi_j(\tau)D^l f(\tau).$$

where $\psi_j(t) = (t-t_{j-m+1})\dots(t-t_j)$. We obtain successively

$$D^m\psi_j(\tau) = (-1)^m m!, \quad D^{m-1}\psi_j(\tau) = (-1)^m m!(\tau-\theta_j),$$

$$D^{m-2}\psi_j(\tau) = \frac{1}{2}(-1)^m m!(\tau^2 - 2\theta_j t + \theta_j^{(2)}).$$

More specifically, taking $\tau = \theta_j$, we get

$$D^{m-1}\psi_j(\theta_j) = 0, \quad D^{m-2}\psi_j(\theta_j) = \frac{1}{2}(-1)^m m!(\theta_j^{(2)} - \theta_j^2)$$

Thus, we can we define the QI exact on $\mathbb{P}_2$

$$Q_2 f = \sum_{j\in J}\lambda_j(f)B_j,$$

whose coefficient functionals are given by

$$\lambda_j(f) = f(\theta_j) - \frac{1}{2}\bar{\theta}_j^{(2)}D^2 f(\theta_j), \ \text{ with } \ \bar{\theta}_j^{(2)} = \theta_j^2 - \theta_j^{(2)}.$$

We recall the expansion (see e.g. [4][15]):

$$\bar{\theta}_j^{(2)} = \frac{1}{m^2(m-1)}\sum_{1\le r<s\le m}(t_{j-r} - t_{j-s})^2.$$

On the other hand, $\frac{1}{2}D^2 f(\theta_j)$ coincides on the space $\mathbb{P}_2$ with the second order divided difference $[\theta_{j-1}, \theta_j, \theta_{j+1}]f$, therefore the dQI defined by

$$Q_2^* f = \sum_{j \in J} \lambda_j^*(f) B_j,$$

with coefficient functionals

$$\lambda_j^*(f) = f(\theta_j) - \bar{\theta}_j^{(2)}[\theta_{j-1}, \theta_j, \theta_{j+1}]f,$$

is also exact on $\mathbb{P}_2$. Moreover, one can write

$$\lambda_j^*(f) = a_j f(\theta_{j-1}) + b_j f(\theta_j) + c_j f(\theta_{j+1}), \quad \text{with}$$

$a_j = -\bar{\theta}_j^{(2)}/\Delta\theta_{j-1}(\Delta\theta_{j-1}+\Delta\theta_j)$, $b_j = 1+\bar{\theta}_j^{(2)}/\Delta\theta_{j-1}\Delta\theta_j$, $c_j = -\bar{\theta}_j^{(2)}/\Delta\theta_j(\Delta\theta_{j-1}+\Delta\theta_j)$. So, according to the introduction

$$\|Q_2^*\|_\infty \leq \max_{j \in J}(|a_j| + |b_j| + |c_j|) \leq 1 + 2 \max_{j \in J} \bar{\theta}_j^{(2)}/\Delta\theta_{j-1}\Delta\theta_j.$$

The following theorem extends a result given for quadratic splines in [11][22][23].

Theorem 1 *For any degree m, the dQIs Q_2^* are uniformly bounded. More specifically, for all partitions of I:*

$$\|Q_2^*\|_\infty \leq [\frac{1}{2}(m+4)].$$

proof: We only give the proof for $m = 2k+1$, the case $m = 2k$ being similar. For the sake of simplicity, we take $j = m$, i.e. we shall determine an upper bound of the ratio

$$N_m/D_m = \bar{\theta}_m^{(2)}/\Delta\theta_{m-1}\Delta\theta_m$$

with

$$N_m = \bar{\theta}_m^{(2)} = \frac{1}{m^2(m-1)} \sum_{1 \leq r < s \leq m} (t_r - t_s)^2.$$

Setting $H = \sum_{i=2}^m h_i$, then we get a lower bound for the denominator

$$D_m = \frac{1}{m^2}(t_m - t_0)(t_{m+1} - t_1) = \frac{1}{m^2}(h_1 + H)(H + h_{m+1}) \geq \frac{H^2}{m^2}.$$

The numerator N_m is composed of k pairs of sums (S_p, S'_p)

$$S_p = \sum_{s-r=p} (t_r - t_s)^2, \quad S'_p = \sum_{s-r=m-p} (t_r - t_s)^2,$$

for $1 \leq p \leq k$. Both sums contain at most p times the terms h_i^2 and $2h_i h_j$ $(i \neq j)$, hence we can write $S_p + S'_p \leq 2pH^2$, which implies

$$N_m \leq \frac{2H^2}{m^2(m-1)}(1+2+\ldots+k) = \frac{(k+1)H^2}{2m^2},$$

so, we get

$$N_m/D_m \leq \frac{1}{2}(k+1),$$

and finally, for $m = 2k+1$ odd

$$\|Q_2^*\|_\infty \leq k+2 = \frac{1}{2}(m+3) = [\frac{1}{2}(m+4)].$$

For $m = 2k$, we obtain respectively $D_m \geq \frac{H^2}{m^2}$ and $N_m \leq \frac{H^2}{4(m-1)}$, whence $N_m/D_m \leq \frac{k^2}{2k-1}$, and finally for $m = 2k$ even

$$\|Q_2^*\|_\infty \leq k+2 = \frac{1}{2}(m+4) = [\frac{1}{2}(m+4)]. \quad \square$$

4 Existence and characterization of near-best discrete quasi-interpolants

4.1 Existence of near-best dQIs

We consider the following family of dQIs defined on $I = \mathbb{R}$ endowed with an arbitrary non-uniform strictly increasing sequence of knots $T = \{t_i,\, i \in \mathbb{Z}\}$,

$$Qf = Q_{p,q}f = \sum_{i\in\mathbb{Z}} \lambda_i(f) B_i.$$

Their coefficient functionals depend on $2p+1$ parameters, with $2p \geq m$,

$$\lambda_i(f) = \sum_{s=-p}^{p} a_i(s) f(\theta_{i+s}),$$

and they are exact on the space $\mathbb{P}_q$, where $q \leq m$. The latter condition is equivalent to $Qe_r = e_r$ for all monomials of degrees $0 \leq r \leq q$. It implies that for all indices i, the parameters $a_i(s)$ satisfy the system of $q+1$ linear equations:

$$\sum_{s=-p}^{p} a_i(s)\theta_{i+s}^r = \theta_i^{(r)}, \quad 0 \leq r \leq q.$$

The Vandermonde matrix $V_i \in \mathbb{R}^{(q+1)\times(2p+1)}$ of this system, with coefficients $V_i(r,s) = \theta_{i+s}^r$, is of maximal rank $q+1$, therefore there are $2p-q$ *free parameters*. Denoting $b_i \in \mathbb{R}^{q+1}$ the vector with components $b_i(r) = \theta_i^{(r)}, \quad 0 \leq r \leq q$, and by $a_i \in \mathbb{R}^{2p+1}$ the vector with components $a_i(s)$, we consider the sequence of minimization problems, for $i \in \mathbb{Z}$:

$$\min \|a_i\|_1, \quad V_i a_i = b_i. \qquad (M_i)$$

We have already seen in the introduction that $\nu_1^*(Q) = \max_{i\in\mathbb{Z}} \min \|a_i\|_1$ is an upper bound of $\|Q\|_\infty$ which is easier to evaluate than the true norm of the dQI.

Theorem 2 *The above minimization problems* (M_i) *have always solutions, which, in general, are non unique.*

proof: The *objective function being convex* and *the domains being affine subspaces*, these classical optimization problems have always solutions, in general non unique. $\square$.

4.2 Characterization of optimal solutions

For $b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$, let us consider the l_1-minimization problem

$$\min \|r(a)\|_1, \quad r(a) = b - Aa. \tag{M}$$

We recall the characterization of optimal solutions for l_1-problems given in [30], chapter 6. Define the sets

$$Z(a) = \{1 \leq i \leq m \,;\, r_i(a) = 0\},$$

$$V(a) = \{v \in \mathbb{R}^m \,;\, \|v\|_\infty \leq 1,\ v_i = \operatorname{sgn}(r_i(a)) \text{ for } i \notin Z(a)\}.$$

Theorem 3 *The vector* $a^* \in \mathbb{R}^n$ *is a solution of (M) if and only if there exists a vector* $v^* \in V(a^*)$ *satisfying* $A^T v^* = 0$.

5 A family of spline discrete quasi-interpolants exact on $\mathbb{P}_2$

In this section, we restrict our study to the subfamily of spline dQIs which are exact on $\mathbb{P}_2$, i.e. we consider the dQIs $Q_p = Q_{p,2}$.
We shall need some set of indices. Let $\overline{K} = \{-p, \ldots, p\}$, and $K^* = \{-p, 0, p\}$. Then, we can write $K := \overline{K} \setminus K^* = K_1 \cup K_2$, where $K_1 = \{-p+1, \ldots, -1\}$, and $K_2 = \{1, \ldots, p-1\}$.
The three equations expressing the exactness of Q_p on $\mathbb{P}_2$ can be written

$$a_i\,(-p) + a_i\,(0) + a_i\,(p) = 1 - \sum_{r \in K} a_i\,(r)$$

$$\theta_{i-p} a_i\,(-p) + \theta_i a_i\,(0) + \theta_{i+p} a_i\,(p) = \theta_i - \sum_{r \in K} \theta_{i+r} a_i\,(r)$$

$$\theta_{i-p}^2 a_i\,(-p) + \theta_i^2 a_i\,(0) + \theta_{i+p}^2 a_i\,(p) = \theta_i^{(2)} - \sum_{r \in K} \theta_{i+r}^2 a_i\,(r)$$

Let $(a_i^*\,(-p)\,, a_i^*\,(0)\,, a_i^*\,(p))$ be the unique solution of the system with the right-hand side obtained by taking $a_i\,(r) = 0$ for all $r \in K$. Using Cramer's rule, we

obtain

$$a_i^*(-p) = -\overline{\theta}_i^{(2)} / (\theta_{i+p} - \theta_{i-p})(\theta_i - \theta_{i-p}),$$
$$a_i^*(0) = 1 + \overline{\theta}_i^{(2)} / (\theta_{i+p} - \theta_i)(\theta_i - \theta_{i-p}),$$
$$a_i^*(p) = -\overline{\theta}_i^{(2)} / (\theta_{i+p} - \theta_{i-p})(\theta_{i+p} - \theta_i).$$

Then we can express the general solution of the above system in the form

$$a_i(-p) = a_i^*(-p) - \sum_{r \in K_1} \alpha_{i,r} a_i(r) + \sum_{s \in K_2} \alpha_{i,s} a_i(s),$$
$$a_i(0) = a_i^*(0) - \sum_{r \in K_1} \beta_{i,r} a_i(r) - \sum_{s \in K_2} \beta_{i,s} a_i(s),$$
$$a_i(p) = a_i^*(p) + \sum_{r \in K_1} \gamma_{i,r} a_i(r) - \sum_{s \in K_2} \gamma_{i,s} a_i(s),$$

with

$$\alpha_{i,j} = \begin{cases} \dfrac{(\theta_i - \theta_{i+j})(\theta_{i+p} - \theta_{i+j})}{(\theta_i - \theta_{i-p})(\theta_{i+p} - \theta_{i-p})}, & \text{if } j \in K_1, \\ \dfrac{(\theta_{i+j} - \theta_i)(\theta_{i+p} - \theta_{i+j})}{(\theta_i - \theta_{i-p})(\theta_{i+p} - \theta_{i-p})}, & \text{if } j \in K_2, \end{cases}$$

$$\gamma_{i,j} = \begin{cases} \dfrac{(\theta_{i+j} - \theta_{i-p})(\theta_i - \theta_{i+j})}{(\theta_{i+p} - \theta_{i-p})(\theta_{i+p} - \theta_i)}, & \text{if } j \in K_1, \\ \dfrac{(\theta_{i+j} - \theta_{i-p})(\theta_{i+j} - \theta_i)}{(\theta_{i+p} - \theta_{i-p})(\theta_{i+p} - \theta_i)}, & \text{if } j \in K_2, \end{cases}$$

and

$$\beta_{i,j} = \frac{(\theta_{i+j} - \theta_{i-p})(\theta_{i+p} - \theta_{i+j})}{(\theta_i - \theta_{i-p})(\theta_{i+p} - \theta_i)}, \ j \in K_1 \cup K_2.$$

We denote by Q_p^* the spline dQI whose coefficient functionals are

$$\lambda_i^*(f) = a_i^*(-p) f(\theta_{i-p}) + a_i^*(0) f(\theta_i) + a_i^*(p) f(\theta_{i+p}).$$

In that case, an upper bound of the norm of this QI is $\max_{i \in \mathbb{Z}} \nu_i^*$, where

$$\nu_i^* = |a_i^*(-p)| + |a_i^*(0)| + |a_i^*(p)|.$$

Theorem 4 *For all $p \geq m$, the infinity norms of the spline dQIs Q_p^* are uniformly bounded by $\frac{m+1}{m-1}$. This bound is independent of p and of the sequence of knots T.*

proof: We have to find a good upper bound of ν_i^*. We recall that $\theta_i = \frac{1}{m} \sum_{r \in \mathbb{N}_m} t_{i+1-r}$ and $\overline{\theta}_i^{(2)} = \frac{1}{m^2(m-1)} S_1$, with

$$S_1 = \sum_{1 \leq r < s \leq m} (t_{i+1-r} - t_{i+1-s})^2.$$

Define the following sums:

$$S_2 = \sum_{r\in\mathbb{N}_m} (t_{i+1-r} - t_{i+1-m-r}) = \sum_{r\in\mathbb{N}_m} \sum_{k=1}^{m} h_{i+2-r-k},$$

$$S_3 = \sum_{r\in\mathbb{N}_m} (t_{i+1+m-r} - t_{i+1-r}) = \sum_{r\in\mathbb{N}_m} \sum_{k=1}^{m} h_{i+1-r+k}.$$

As $p \geq m$, we obtain

$$\theta_i - \theta_{i-p} = \frac{1}{m} \sum_{r\in\mathbb{N}_m} (t_{i+1-r} - t_{i+1-p-r}) = \frac{1}{m} \sum_{r\in\mathbb{N}_m} \sum_{k=1}^{p} h_{i+2-r-k} \geq \frac{1}{m} S_2,$$

$$\theta_{i+p} - \theta_i = \frac{1}{m} \sum_{r\in\mathbb{N}_m} (t_{i+1+p-r} - t_{i+1-r}) = \frac{1}{m} \sum_{r\in\mathbb{N}_m} \sum_{k=1}^{p} h_{i+1+k-r} \geq \frac{1}{m} S_3.$$

The proof being essentially the same for all $i \in \mathbb{Z}$, we can restrict our study to the case $i = m$. In that case, we get

$$S_2 = mh_1 + \sum_{k=1}^{m-1} k\,(h_{m+1-k} + h_{k+1-m}) \geq S_2' = (m-1)\,h_2 + \cdots + 2h_{m-1} + h_m,$$

$$S_3 = mh_{m+1} + \sum_{k=1}^{m-1} k\,(h_{2m+1-k} + h_{k+1}) \geq S_3' = (m-1)\,h_m + \cdots + 2h_3 + h_2.$$

Setting $H_k = h_2 + \cdots + h_{k+1}$, for $1 \leq k \leq m-1$, and $H = H_{m-1}$ for short as in the proof of theorem 1, we have

$$S_2' = \sum_{i=0}^{m-1} H_i, \quad S_3' = H + \sum_{i=1}^{m-2} (H - H_i) = mH - S_2',$$

whence

$$S_2 S_3 \geq S_2' S_3' = mH \sum_{i=0}^{m-1} H_i - \left(\sum_{i=0}^{m-1} H_i \right)^2 .$$

Now, we come back to S_1 and we shall prove that $S_1 \leq S_2' S_3' \leq S_2 S_3$. S_1 can be written under the form

$$S_1 = \sum_{r=1}^{m-1} \sum_{j=1}^{m-r} (h_{r+1} + \cdots + h_{r+j})^2 = \sum_{i=1}^{m-1} H_i^2 + \sum_{j=1}^{m-2} \sum_{i=j+1}^{m-1} (H_i - H_j)^2 ,$$

from which we deduce

$$S_1 = (m-1) \sum_{i=1}^{m-1} H_i^2 - 2 \sum_{j=1}^{m-2} H_j \sum_{i=j+1}^{m-1} H_i = m \sum_{i=1}^{m-1} H_i^2 - \left(\sum_{i=1}^{m-1} H_i \right)^2 .$$

Then we use the fact that, for all $1 \leq i \leq m-1$, $H_i \leq H$, whence $H_i^2 \leq H\ H_i$ and $\sum_{i=0}^{m-1} H_i^2 \leq H \sum_{i=0}^{m-1} H_i$. So, we obtain

$$S_1 \leq mH \sum_{i=0}^{m-1} H_i - \left(\sum_{i=0}^{m-1} H_i \right)^2 = S_2' S_3' \leq S_2 S_3.$$

Finally, for all $i \in \mathbb{Z}$, we have

$$\nu_i^* = 1 + \frac{2}{m-1} \frac{S_1}{S_2 S_3} \leq 1 + \frac{2}{m-1} = \frac{m+1}{m-1},$$

whence $\left\| Q_p^* \right\|_\infty \leq \max_{i \in \mathbb{Z}} \nu_i^* \leq \dfrac{m+1}{m-1}$. □

In the next section we prove that the quasi-interpolants Q_p^* are near-best in the sense of section 4 under some additional conditions on the partitions.

6 The family Q_p^* of discrete quasi-interpolants is near-best

Let us write the minimization problem (P_d) of section 4 in Watson's form. Taking into account the expression of the solution a_i of the system equivalent to the exactness on $\mathbb{P}_2$ of $Q_{p,2}$, we can write

$$\|a_i\|_1 = \|a_i^* - A_i \widetilde{a}_i\|,$$

where

$$\widetilde{a}_i = \left(a_i(-p+1), \ldots, a_i(-1), a_i(1), \ldots, a_i(p-1)\right)^T \in \mathbb{R}^{2p-2},$$
$$a_i^* = \left(a_i^*(-p), 0, \ldots, 0, a_i^*(0), 0, \ldots, a_i^*(p)\right)^T \in \mathbb{R}^{2p+1},$$

and $A_i \in \mathbb{R}^{(2p+1)\times(2p-2)}$ is given by

$$A_i = \begin{pmatrix} \alpha_{i,-p+1} & \cdots & \alpha_{i,-1} & -\alpha_{i,-1} & \cdots & \alpha_{i,p-1} \\ -1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -1 & 0 & \cdots & 0 \\ \beta_{i,-p+1} & \cdots & \beta_{i,-1} & \beta_{i,1} & \cdots & \beta_{i,p-1} \\ 0 & \cdots & 0 & -1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & -1 \\ -\gamma_{i,-p+1} & \cdots & -\gamma_{i,-1} & \gamma_{i,1} & \cdots & \gamma_{i,p-1} \end{pmatrix}.$$

Theorem 5 *Assume that the sequence of knots T satisfies, for all $i \in \mathbb{Z}$, the following properties*

$$\theta_{i-1} + \theta_i \leq \theta_{i-p} + \theta_p \leq \theta_i + \theta_{i+1},$$

then, for all $i \in \mathbb{Z}$, a_i^* is an optimal solution of the local minimization problem (M_i). Thus, for all $p \geq m$, the spline dQIs Q_p^* (theorem 4) are near-best.

proof: According to theorem 3, we must find a vector $v^* \in \mathbb{R}^{2p+1}$ satisfying

$$\|v^*\|_\infty \leq 1, \quad A_i^T v^* = 0, \quad v^*(r) = \operatorname{sgn}(a_i^*(r)) \text{ for } r = -p, 0, p.$$

Let us choose $v^*(-p) = -1$, $v^*(0) = 1$, $v^*(p) = -1$, and

$$v^*(j) = \begin{cases} -\alpha_{i,r} + \beta_{i,r} + \gamma_{i,r}, & \text{if } j \in K_1, \\ \alpha_{i,r} + \beta_{i,r} - \gamma_{i,r}, & \text{if } j \in K_2. \end{cases}$$

Then it is easy to verify that the equations $A_i^T v^* = 0$ are satisfied. Moreover, the above expressions of $a_i^*(r)$ for $r = -p, 0, p$, with $\overline{\theta}_i^{(2)} > 0$ imply that $v^*(r) = \operatorname{sgn}(a_i^*(r))$. It only remains to prove that $|v^*(j)| \leq 1$ for $j \in K_1 \cup K_2$. As $\beta_{i,r} = 1 - \alpha_{i,r} + \gamma_{i,r}$ for $r \in K_1$ and $\beta_{i,s} = 1 + \alpha_{i,s} - \gamma_{i,s}$ for $s \in K_2$, it is equivalent to prove

$$0 \leq \alpha_{i,r} - \gamma_{i,r} \leq 1, \quad 0 \leq \gamma_{i,s} - \alpha_{i,s} \leq 1, \quad \text{for } (r,s) \in K_1 \times K_2.$$

We only detail the proof for $r \in K_1$, that for $s \in K_2$ being quite similar. Using the expressions of $\alpha_{i,r}$ and $\gamma_{i,r}$ given in section 5, we get

$$\alpha_{i,r} - \gamma_{i,r} = \frac{(\theta_i - \theta_{i+r})\left[(\theta_{i+p} + \theta_{i-p}) - (\theta_{i+r} + \theta_i)\right]}{(\theta_i - \theta_{i-p})(\theta_{i+p} - \theta_i)},$$

and we shall have $\alpha_{i,r} - \gamma_{i,r} \geq 0$ if and only if

$$\theta_{i+r} + \theta_i \leq \theta_{i+p} + \theta_{i-p}$$

for all $r \in K_1$. However, since we have $\theta_{i+r} + \theta_i \leq \theta_{i-1} + \theta_i$, there only remains the unique condition

$$\theta_{i-1} + \theta_i \leq \theta_{i-p} + \theta_{i+p}.$$

The other inequality $\alpha_{i,r} - \gamma_{i,r} \leq 1$ can be written

$$(\theta_i - \theta_{i+r})\left[(\theta_{i+p} + \theta_{i-p}) - (\theta_{i+r} + \theta_i)\right] \leq (\theta_i - \theta_{i-p})(\theta_{i+p} - \theta_i).$$

Setting $\delta_1 = \theta_{i+r} - \theta_{i-p}$, $\delta_2 = \theta_i - \theta_{i+r}$, and $\delta_3 = \theta_{i+p} - \theta_i$, the latter inequality can be written $\delta_2(\delta_3 - \delta_1) \leq \delta_3(\delta_2 + \delta_1)$, or equivalently $\delta_1(\delta_2 + \delta_3) \geq 0$ which is obviously satisfied. For $s \in K_2$, the inequalities $0 \leq \gamma_{i,s} - \alpha_{i,s} \leq 1$ are satisfied if and only if

$$\theta_{i-p} + \theta_{i+p} \leq \theta_i + \theta_{i+1},$$

whence the conditions on the sequence of knots. $\square$

Remark 1 *Theorem 5 imposes some additional conditions on the sequence of knots. For quadratic splines, we have studied arithmetic and geometric sequences: in both cases, the higher is p, the stronger are the conditions and, for $p \to +\infty$, T is closer and closer to a uniform sequence.*

Remark 2 *Even if the partition T does not satisfy the hypotheses of theorem 5, the operator Q_p^* is still a good QI because its infinity norm is small and uniformly bounded.*

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7 Uniformly bounded integral quasi-interpolants of Goodman-Sharma type

General integral spline quasi-interpolants (iQIs) already appear in [4]-[5][14][21][26]. Here we have chosen to study a family of QIs which we call Goodman-Sharma (GS-) type iQIs, as they first appear in [10]. They seem simpler and more interesting than those studied in [21] and [26]. In the case of splines of degree m on a partition in n subintervals of a bounded interval $I = [a, b]$, the simplest GS-type iQI can be written as follows:

$$G_1 f = f(t_0) B_0 + \sum_{i=1}^{n+m-2} \lambda_i(f) B_i + f(t_n) B_{n+m-1},$$

where the integral coefficient functionals are defined by

$$\lambda_i(f) = \int_a^b M_i(t) f(t) \, dt = \langle M_i, f \rangle,$$

M_i being the B-spline of degree $m-2$ with support $\Sigma_i = [t_{i-m+1}, t_i]$, normalized by $\lambda_i(e_0) = \mu_i^{(0)} = \int_{\mathbb{R}} M_i = 1$. It is easy to verify that G_1 is exact on P_1 and that $\|G_1\|_\infty = 1$. In this section, we shall study the family of GS-type iQIs defined by

$$G_2 f = f(t_0) B_0 + \sum_{i=1}^{n+m-2} [a_i \lambda_{i-1}(f) + b_i \lambda_i(f) + c_i \lambda_{i+1}(f)] B_i + f(t_n) B_{n+m-1},$$

which we want to be exact on $\mathbb{P}_2$. The three constraints $G_2 e_k = e_k$, $k = 0, 1, 2$, lead to the following system of equations, for each $1 \le i \le n+m-2$:

$$a_i + b_i + c_i = 1, \ \theta_{i-1} a_i + \theta_i b_i + \theta_{i+1} c_i = \theta_i, \ \mu_{i-1}^{(2)} a_i + \mu_i^{(2)} b_i + \mu_{i+1}^{(2)} c_i = \theta_i^{(2)}.$$

We recall the values of the first moments of M_i:

$$\mu_i^{(1)} = \frac{1}{m} \sum_{1 \le r \le m} t_{i+1-r} = \frac{1}{m} \sum_{1 \le r \le m} t_{i-m+r} = \theta_i,$$

$$\mu_i^{(2)} = \frac{2}{m(m+1)} \sum_{1 \le r \le s \le m} t_{i+1-r} t_{i+1-s} = \frac{2}{m(m+1)} \sum_{1 \le r \le s \le m} t_{i-m+r} t_{i-m+s}.$$

Theorem 6 *For any degree m, the iQIs G_2 are uniformly bounded indepently of the partition of I. For $m = 2k$ or $2k+1$, there holds*

$$\|G_2\|_\infty \le 2k + 3.$$

proof: Taking the differences of the second and third equations above ($G_2 e_k = e_k$, $k = 1, 2$) with the first one ($G_2 e_0 = e_0$) times θ_i and $\mu_i^{(2)}$ resp., we get

$$(\theta_i - \theta_{i-1}) a_i = (\theta_{i+1} - \theta_i) c_i, \quad \left(\mu_i^{(2)} - \mu_{i-1}^{(2)}\right) a_i + \left(\mu_i^{(2)} - \mu_{i+1}^{(2)}\right) c_i = \mu_i^{(2)} - \theta_i^{(2)}.$$

Setting $a_i = (\theta_{i+1} - \theta_i)\,\alpha_i$ and $c_i = (\theta_i - \theta_{i-1})\,\alpha_i$, we obtain

$$\left[\Delta\mu^{(2)}_{i-1}\Delta\theta_i - \Delta\mu^{(2)}_i\Delta\theta_{i-1}\right]\alpha_i = \mu^{(2)}_i - \theta^{(2)}_i. \tag{E}$$

Using the expressions of the various coefficients in terms of symmetric functions of the knots, we obtain the following form for the coefficient of α_i in equation (E)

$$\frac{2}{m^2\,(m+1)}\left[(t_{i+1} - t_{i+1-m})\,\Delta\overline{\sigma}_2\,(T_{i-1}) - (t_i - t_{i-m})\,\Delta\overline{\sigma}_2\,(T_i)\right].$$

Setting $l_i = t_i - t_{i+1-m}$, we can write $t_i - t_{i-m} = h_{i+1-m} + l_i$, $t_{i+1} - t_{i-m} = h_{i+1-m} + l_i + h_{i+1}$ and $t_{i+1} - t_{i+1-m} = l_i + h_{i+1}$. Then we have

$$\Delta\overline{\sigma}_2\,(T_i) = (t_{i+1} - t_{i+1-m})\sum_{r=i-m+1}^{i+1} t_r,$$

and the coefficient of α_i in equation (E) is given by

$$\begin{aligned}\Delta\mu^{(2)}_{i-1}\Delta\theta_i - \Delta\mu^{(2)}_i\Delta\theta_{i-1} &= -\,(t_{i+1} - t_{i+1-m})\,(t_i - t_{i-m})\,(t_{i+1} - t_{i-m})\\ &= -\,(l_i + h_{i+1})\,(h_{i+1-m} + l_i)\,(h_{i+1-m} + l_i + h_{i+1})\,.\end{aligned}$$

Now, writting $\overline{\sigma}_2\,(T_i) = \sigma_2\,(T_i) + \sum_{r=i+1-m}^{i} t_r^2$, we get successively

$$\begin{aligned}\mu^{(2)}_i - \theta^{(2)}_i &= \frac{2}{m\,(m+1)}\overline{\sigma}_2\,(T_i) - \frac{2}{m\,(m-1)}\sigma_2\,(T_i)\\ &= \frac{2}{m\,(m^2-1)}\left((m-1)\,\overline{\sigma}_2\,(T_i) - (m+1)\,\sigma_2\,(T_i)\right)\\ &= \frac{2}{m\,(m^2-1)}\left((m-1)\sum_{r=i+1-m}^{i} t_r^2 - 2\sigma_2\,(T_i)\right)\\ &= \frac{2}{m\,(m^2-1)}\sum_{i+1-m\le r<s\le i}(t_r - t_s)^2\,.\end{aligned}$$

Setting $\omega_i = \sum_{i+1-m\le r<s\le i}(t_r - t_s)^2$, we obtain

$$\alpha_i = -\frac{m}{m-1}\,\frac{\omega_i}{(h_{i+1-m} + l_i)\,(h_{i+1-m} + l_i + h_{i+1})\,(l_i + h_{i+1})},$$

from which we deduce

$$\begin{aligned}a_i &= -\frac{1}{m-1}\,\frac{\omega_i}{(h_{i+1-m} + l_i)\,(h_{i+1-m} + l_i + h_{i+1})},\\ c_i &= -\frac{1}{m-1}\,\frac{\omega_i}{(h_{i+1-m} + l_i + h_{i+1})\,(l_i + h_{i+1})},\end{aligned}$$

and

$$b_i = 1 - a_i - c_i, \quad |a_i| + |b_i| + |c_i| = 1 + 2\,(|a_i| + |c_i|)\,.$$

On the other hand, we have

$$|a_i| + |c_i| = \frac{1}{m-1} \frac{\omega_i \left(h_{i+1-m} + 2l_i + h_{i+1}\right)}{\left(h_{i+1-m} + l_i\right)\left(h_{i+1-m} + l_i + h_{i+1}\right)\left(l_i + h_{i+1}\right)}.$$

As $\omega_i \leq k\,(k+1)\, l_i^2$ for $m = 2k+1$ (see proof of theorem 1 with respect the upper bound for N_m), we obtain

$$\omega_i \leq k\,(k+1)\left(h_{i+1-m} + l_i\right)\left(l_i + h_{i+1}\right).$$

Moreover, it is obvious that

$$h_{i+1-m} + 2l_i + h_{i+1} \leq 2\left(h_{i+1-m} + l_i + h_{i+1}\right).$$

Therefore, we finally obtain for all i

$$|a_i| + |c_i| \leq \frac{2k\,(k+1)}{m-1} = k+1 = \frac{1}{2}\,(m+1)\,,$$

whence the uniform upper bound for the norm of G_2

$$\|G_2\|_\infty \leq m+2,$$

which is both independent of the partition and of the (odd) degree of the spline. For $m = 2k$ even, a similar computation leads to the uniform bound

$$\|G_2\|_\infty \leq m+3.$$

□

8 Existence and characterization of near-best integral quasi-interpolants

Now, we consider the family of iQIs

$$G_{p,q} f = \sum_{i \in \mathbb{Z}} \lambda_i\,(f)\, B_i,$$

whose coefficient functionals depend on $2p+1$ parameters:

$$\lambda_i\,(f) = \sum_{s=-p}^{p} a_i\,(s) \int_{\Sigma_{i+s}} M_{i+s}\,(t)\, f\,(t)\, dt,$$

and which are exact on $\mathbb{P}_q$. As in section 7, M_j is the B-spline of degree $m-2$, with support $\Sigma_j = [t_{j+1-m}, t_j]$ normalized by $\int_{\mathbb{R}} M_j = 1$. The constraints $G_{p,q}\,(e_r) = e_r$ are equivalent to the following systems of linear equations, for all $i \in \mathbb{Z}$:

$$\sum_{s=-p}^{p} \mu_{i+s}^{(r)} a_i\,(s) = \theta_i^{(r)},\ 0 \leq r \leq q,$$

whose matrix coefficients $W_i \in \mathbb{R}^{(q+1)\times(2p+1)}$, defined by $W_i(r,s) = \mu_{i+s}^{(r)}$, is of maximal rank $q+1$ (see e.g. [26]), thus there remains $2p-q$ free parameters. In view of the introduction, as $\max_{i\in\mathbb{Z}} \|a_i\|_1$ is an upper bound of the true infinity norm of the iQI, we want to solve the minimization problems, for all i:

$$\min \|a_i\|_1, \quad W_i a_i = b_i. \qquad (\overline{M}_i)$$

As in section 4.1, the objective function being convex and the domains being affine subspaces, we can conclude

Theorem 7 *The above minimization problems* $(\overline{M}_i)$ *have always solutions, in general non unique.*

As in section 4.2, we shall use the characterizacion of optimal solutions for l_1-problems (M) given in [30], chapter 6.

9 A family of integral spline quasi-interpolants exact on $\mathbb{P}_2$

In this section, we restrict our study to the subfamily $q = 2$ of the above spline iQIs which are exact on $\mathbb{P}_2$. Moreover, we assume that $p \geq m$ in order to insure that the three sets of knots T_{i-p}, T_i, and T_{i+p} are pairwise disjoint. Now, the matrix coefficients of the linear system equivalent to the exactness of $G_{p,2}$ on $\mathbb{P}_2$ is of maximal rank 3, and we have $2p-2$ free parameters. Let us denote by G_p^* the spline iQI whose coefficient functionals are

$$\lambda_i^*(f) = a_i^*(-p)\langle M_{i-p}, f\rangle + a_i^*(0)\langle M_i, f\rangle + a_i^*(p)\langle M_{i+p}, f\rangle,$$

where

$$\begin{aligned} a_i^*(-p) + a_i^*(0) + a_i^*(p) &= 1, \\ \theta_{i-p} a_i^*(-p) + \theta_i a_i^*(0) + \theta_{i+p} a_i^*(p) &= \theta_i, \\ \mu_{i-p}^{(2)} a_i^*(-p) + \mu_i^{(2)} a_i^*(0) + \mu_{i+p}^{(2)} a_i^*(p) &= \theta_i^{(2)}, \end{aligned}$$

that is, their coefficients are the unique solution of the system obtained by taking $a_i(r) = 0$ for all $r \in K$ (we use the same notations as in section 5).
The following result is the analog of theorem 4 for iQIs.

Theorem 8 *For any degree $m \geq 2$, and for all $p \geq m$, the iQIs G_p^* are uniformly bounded independently of the partition of I. More specifically, there holds*

$$\|G_p^*\|_\infty \leq 1 + \frac{1}{4} C(m), \quad \textit{with} \quad C(m) = \begin{cases} \frac{m^2(m+2)}{(m-1)^2} & \textit{for } m \textit{ even}, \\ \frac{(m+1)^2}{m-1} & \textit{for } m \textit{ odd}. \end{cases}$$

proof: We shall prove that $|a_i^*(-p)| + |a_i^*(0)| + |a_i^*(p)| \leq 1 + \frac{1}{4}C(m)$ for all $i \in \mathbb{Z}$, which is sufficient to insure the result. For the sake of simplicity, we can assume that $i = m$. Solving the corresponding linear system, we get

$$a_m^*(-p) = -\frac{\xi_m \Delta_p \theta_m}{\delta_m}, \quad a_m^*(p) = -\frac{\xi_m \Delta_p \theta_{m-p}}{\delta_m}, \quad a_0^*(0) = 1 - a_m^*(-p) - a_m^*(p),$$

where

$$\xi_m = \mu_m^{(2)} - \theta_m^{(2)}, \quad \text{and} \quad \delta_m = \Delta_p \theta_{m-p} \Delta_p \mu_m^{(2)} - \Delta_p \theta_m \Delta_p \mu_{m-p}^{(2)},$$

with

$$\Delta_p \theta_l = \theta_{l+p} - \theta_l, \quad \Delta_p \mu_l^{(2)} = \mu_{l+p}^{(2)} - \mu_l^{(2)}, \quad l = m, m-p.$$

As proved in theorem 6, we have

$$\xi_m = \frac{2}{m(m^2-1)} \omega_m = \frac{2}{m(m^2-1)} \sum_{1 \leq r < s \leq m} (t_r - t_s)^2.$$

For the expression of δ_m, we need some additional notations. For $1 \leq i \leq m$, we define

$$l_i = t_1 - t_{i-p}, \quad l_i' = t_i - t_{i-p} = \sum_{r=2}^{i} h_r + l_i.$$

Similarly, for $1 \leq j \leq m$, we define

$$l_{m+j} = t_{m+p} - t_j, \quad l_{m+j}' = t_{j+p} - t_j = l_{m+j} + \sum_{s=j}^{m-1} h_{s+1}.$$

Taking into account the definitions of θ_i and $\mu_i^{(2)}$ for $i = m-p, m, m+p$, we obtain after some algebraic calculations the following expressions:

$$\begin{aligned}
\Delta_p \theta_{m-p} &= \frac{1}{m} \sum_{i=1}^{m} l_i', \quad \Delta_p \theta_m = \frac{1}{m} \sum_{j=1}^{m} l_{m+j}', \\
\Delta_p \mu_{m-p}^{(2)} &= \frac{2}{m(m+1)} \sum_{i=1}^{m} l_i' \left(\sum_{r=i}^{m} t_{r-p} + \sum_{s=1}^{i} t_s \right), \\
\Delta_p \mu_m^{(2)} &= \frac{2}{m(m+1)} \sum_{j=1}^{m} l_{m+j}' \left(\sum_{r=j}^{m} t_r + \sum_{s=1}^{j} t_{p+s} \right).
\end{aligned}$$

Now we can write

$$a_m^*(-p) = -\frac{1}{m-1} \frac{\omega_m}{D_m} \sum_{j=1}^{m} l_{m+j}', \quad a_m^*(p) = -\frac{1}{m-1} \frac{\omega_m}{D_m} \sum_{i=1}^{m} l_i',$$

where D_m is equal to

$$\left(\sum_{i=1}^{m} l'_i\right)\left(\sum_{j=1}^{m} l'_{m+j}\left(\sum_{r=j}^{m} t_r+\sum_{s=1}^{j} t_{p+s}\right)\right)-\left(\sum_{j=1}^{m} l'_{m+j}\right)\left(\sum_{i=1}^{m} l'_i\left(\sum_{r=i}^{m} t_{r-p}+\sum_{s=1}^{i} t_s\right)\right)$$

$$=\sum_{i=1}^{m}\sum_{j=1}^{m} l'_i l'_{m+j}\left(\sum_{r=j}^{m} t_r+\sum_{s=1}^{j} t_{p+s}-\sum_{r=i}^{m} t_{r-p}-\sum_{s=1}^{i} t_s\right).$$

Therefore, as we shall see later (p.18) than $D_m>0$, we obtain

$$|a^*_m(-p)|+|a^*_m(p)|=\frac{1}{m-1}\frac{\omega_m}{D_m}\left(\sum_{j=1}^{m} l'_{m+j}+\sum_{i=1}^{m} l'_i\right)=\frac{1}{m-1}\frac{N_m}{D_m},$$

where

$$N_m=\left(\sum_{k=1}^{2m} l'_k\right)\omega_m=\left(\sum_{k=1}^{2m} l'_k\right)\sum_{1\le r<s\le m}(t_r-t_s)^2.$$

Let us compute an upper bound for N_m. Let $H=\sum_{i=2}^{m} h_i$, as in the proof of theorem 1. In the first sum, we have $l'_i+l'_{m+i}=l_i+l_{i+m}+H$, for $1\le i\le m$, so we can write

$$\sum_{k=1}^{2m} l'_k=\sum_{k=1}^{2m} l_k+mH.$$

For ω_m, we have already seen (also in the proof of theorem 1) that

$$\omega_m\le c(m)\,H^2,$$

where $c(m)=k^2$ for $m=2k$ and $c(m)=k(k+1)$ for $m=2k+1$. So, we finally obtain the following upper bound for N_m:

$$N_m\le c(m)\,H^2\left(\sum_{k=1}^{2m} l_k+mH\right).$$

Now, we will compute a lower bound for D_m. Let

$$L_{i,m+j}=\sum_{r=j}^{m} t_r+\sum_{s=1}^{j} t_{p+s}-\sum_{r=i}^{m} t_{r-p}-\sum_{s=1}^{i} t_s$$

be the coefficient of $l'_i l'_{m+j}$ in the double sum defining D_m. Therefore we have

$$D_m\ge l'_m l'_{m+1}L_{m,m+1}+\sum_{i=1}^{m-1} l'_i l'_{m+1}L_{i,m+1}+\sum_{j=2}^{m} l'_m l'_{m+j}L_{m,m+j}.$$

We first observe that

$$l'_m l'_{m+1}L_{m,m+1}=(l_m+H)(l_{m+1}+H)(l_m+l_{m+1}+H)\ge 2H^2(l_m+l_{m+1})+H^3.$$

Then, we obtain successively, for $1 \leq i \leq m-1$,

$$L_{i,m+1} = \sum_{r=i+1}^{m} t_r + t_{p+1} - \sum_{r=i}^{m} t_{r-p} \geq (t_m - t_{m-1-p}) + (t_{p+1} - t_{m-p}) \geq 2H.$$

Similarly, for $2 \leq j \leq m$,

$$L_{m,m+j} = \sum_{s=1}^{j} t_{p+s} - t_{m-p} - \sum_{r=1}^{j-1} t_r \geq (t_{p+1} - t_{m-p}) + (t_{p+2} - t_1) \geq 2H.$$

On the other hand, we also have successively

$$l'_i l'_{m+1} = (l_i + \sum_{r=2}^{i} h_r)(l_{m+1} + H) \geq H(l_i + \sum_{r=2}^{i} h_r),$$

$$l'_m l'_{m+j} = (l_m + H)(l_{m+j} + \sum_{s=j+1}^{m} h_s) \geq H(l_{m+j} + \sum_{s=j+1}^{m} h_s).$$

From these inequalities, we deduce

$$D_m \geq H^3 + 2H^2 \left(\sum_{k=1}^{2m} l_k + \sum_{i=2}^{m-1} \sum_{r=2}^{i} h_r + \sum_{j=2}^{m-1} \sum_{s=j+1}^{m-1} h_s \right).$$

Now, it is easy to see that

$$\sum_{i=2}^{m-1} \sum_{r=2}^{i} h_r + \sum_{j=2}^{m-1} \sum_{s=j+1}^{m-1} h_s = (m-2)H,$$

therefore we obtain the lower bound

$$D_m \geq 2H^2 \left(\sum_{k=1}^{2m} l_k + (m - \frac{3}{2})H \right).$$

Thus, setting $\mathcal{L} = \sum_{k=1}^{2m} l_k$, we have the two inequalities

$$D_m \geq 2H^2 \left(\mathcal{L} + \left(m - \frac{3}{2} \right) H \right), \quad N_m \leq c\,(m)\, H^2 \left(\mathcal{L} + mH \right),$$

from which we deduce

$$\frac{N_m}{D_m} \leq \frac{1}{2} c(m) \frac{\mathcal{L} + mH}{\mathcal{L} + \left(m - \frac{3}{2} \right) H}.$$

For $m \geq 2$ even, it is easy to verify that $m(m-1) \leq (m+2)(m-3/2)$, whence

$$\frac{\mathcal{L} + mH}{\mathcal{L} + \left(m - \frac{3}{2} \right) H} \leq \frac{m+2}{m-1}.$$

For $m \geq 3$ odd, one can verify that $m(m-1) \leq (m+1)(m-3/2)$, whence

$$\frac{\mathcal{L} + mH}{\mathcal{L} + \left(m - \frac{3}{2}\right) H} \leq \frac{m+1}{m-1}.$$

Finally, as $\|G_p^*\|$ is bounded above by

$$|a_m^*(-p)| + |a_m^*(0)| + |a_m^*(p)| = 1 + 2\left(|a_m^*(-p)| + |a_m^*(p)|\right) \leq 1 + \frac{2}{m-1}\frac{N_m}{D_m},$$

(the same upper bound is valid for $|a_i^*(-p)| + |a_i^*(0)| + |a_i^*(p)|$, for all $i \in \mathbb{Z}$), we obtain respectively

$$\|G_p^*\| \leq 1 + \frac{(m+2)c(m)}{(m-1)^2} \text{ for } m \text{ even}$$

$$\|G_p^*\| \leq 1 + \frac{(m+1)c(m)}{(m-1)^2} \text{ for } m \text{ odd}$$

As $c(m) = \frac{1}{4}m^2$ for m even and $c(m) = \frac{1}{4}(m^2-1)$ for m odd, we obtain

$$\|G_p^*\| \leq 1 + \frac{1}{4}C(m)$$

with $C(m) = \dfrac{m^2(m+2)}{(m-1)^2}$ for m even and $C(m) = \dfrac{(m+1)^2}{m-1}$ for m odd, which proves the theorem. $\square$

10 The family G_p^* of integral quasi-interpolants is near-best

We follow the notations and techniques used in section 5 for discrete QIs. As the linear system satisfied by the coefficients of $\lambda_i(f)$

$$\sum_{s=-p}^{p} \mu_{i+s}^{(r)} a_i(s) = \theta_i^{(r)}, \; 0 \leq r \leq 2,$$

is of maximal rank, it can be written as follows

$$a_i(-p) + a_i(0) + a_i(p) = 1 - \sum_{r \in K} a_i(r),$$

$$\theta_{i-p} a_i(-p) + \theta_i a_i(0) + \theta_{i+p} a_i(p) = \theta_i - \sum_{r \in K} \theta_{i+r} a_i(r),$$

$$\mu_{i-p}^{(2)} a_i(-p) + \mu_i^{(2)} a_i(0) + \mu_{i+p}^{(2)} a_i(p) = \theta_i^{(2)} - \sum_{r \in K}^{2} \mu_{i+r}^{(2)} a_i(r),$$

and its general solution is

$$
\begin{aligned}
a_i(-p) &= a_i^*(-p) - \sum_{r\in K_1} \alpha_{i,r} a_i(r) + \sum_{s\in K_2} \alpha_{i,s} a_i(s),\\
a_i(0) &= a_i^*(0) - \sum_{r\in K_1} \beta_{i,r} a_i(r) - \sum_{s\in K_2} \beta_{i,s} a_i(s),\\
a_i(p) &= a_i^*(p) + \sum_{r\in K_1} \gamma_{i,r} a_i(r) - \sum_{s\in K_2} \gamma_{i,s} a_i(s),
\end{aligned}
$$

where $a_i^*(-p)$, $a_i^*(0)$, and $a_i^*(p)$ are the coefficients of G_p^* studied in section 9 above, and the various coefficients are quotiens of determinants. Specifically,

$$
\begin{aligned}
&\alpha_r = W(r,0,p)/W,\ \beta_r = W(-p,r,p)/W,\ \gamma_r = W(-p,r,0)/W,\\
&\alpha_s = W(0,s,p)/W,\ \beta_s = W(-p,s,p)/W,\ \gamma_s = W(-p,0,s)/W,
\end{aligned}
$$

where $W(k,l,m)$, $k<l<m$, is the determinant with columns $\left(1,\theta_{i+k},\mu_{i+k}^{(2)}\right)^T$, $\left(1,\theta_{i+l},\mu_{i+l}^{(2)}\right)^T$, and $\left(1,\theta_{i+m},\mu_{i+m}^{(2)}\right)^T$, and $W = W(-p,0,p)$. As in section 5, we can write the minimization problem $\mathrm{P_i}$ described in section 8 (with $q=2$) in Watson's form, so we have to minimize $\|a_i\|_1 = \|a_i^* - A_i\widetilde{a}_i\|_1$, where we have used the same notations as in section 5.
According to theorem 3, we must find a vector $v^* \in \mathbb{R}^{2p+1}$ satisfying

$$
\|v^*\|_\infty \le 1, \quad A_i^T v^* = 0, \quad v^*(r) = \operatorname{sgn}(a_i^*(r)) \text{ for } r = -p, 0, p.
$$

Let us choose $v^*(-p) = -1$, $v^*(0) = 1$, $v^*(p) = -1$, and

$$
v^*(j) = \begin{cases} -\alpha_{i,r} + \beta_{i,r} + \gamma_{i,r}, & \text{if } j \in K_1,\\ \alpha_{i,r} + \beta_{i,r} - \gamma_{i,r}, & \text{if } j \in K_2. \end{cases}
$$

Equations $A_i^T v^* = 0$ are satisfied. Moreover, as proved in theorem 6, $\mu_i^{(2)} - \theta_i^{(2)} > 0$ holds, and the explicit expressions for $a_i^*(-p)$, $a_i^*(0)$, and $a_i^*(p)$, similar to those obtained for $i = m$ in the proof of the theorem 8, imply that $v^*(r) = \operatorname{sgn}(a_i^*(r))$. It only remains to prove that, for $(r,s) \in K_1 \times K_2$

$$
|v^*(r)| = |-\alpha_{i,r} + \beta_{i,r} + \gamma_{i,r}| \le 1, \quad |v^*(s)| = |\alpha_{i,s} + \beta_{i,s} - \gamma_{i,s}| \le 1.
$$

As $\beta_{i,r} = 1 - \alpha_{i,r} + \gamma_{i,r}$ for $r \in K_1$ and $\beta_{i,s} = 1 + \alpha_{i,s} - \gamma_{i,s}$ for $s \in K_2$, it is equivalent to prove

$$
0 \le \alpha_{i,r} - \gamma_{i,r} \le 1, \quad 0 \le \gamma_{i,s} - \alpha_{i,s} \le 1, \quad \text{for } (r,s) \in K_1 \times K_2.
$$

We only detail the proof for $i = m$ and for $r \in K_1$, that for $s \in K_2$ being quite similar.
We prove that $\gamma_{m,r} \le \alpha_{m,r}$, $r \in K_1$ by stating that $\gamma_{m,-r} \le \alpha_{m,-r}$, for $r \in \{1,\ldots,p-1\}$, the latter being equivalent to $W(-p,-r,0) \le W(-r,0,p)$. By expanding the various determinants involved, we obtain the inequality

$$
\frac{\mu_m^{(2)} - \mu_{m-r}^{(2)}}{\theta_m - \theta_{m-r}} \le \frac{\mu_{m+p}^{(2)} - \mu_{m-p}^{(2)}}{\theta_{m+p} - \theta_{m-p}}.
$$

For $1 \le i \le m$, let $w'_i = \sum_{j=i+1-r}^{i} h_j,\ w' = \sum_{j=1}^{m} w'_i,\ \tau'_i = \frac{1}{m+1}\left(\sum_{j=i-r}^{m-r} t_j + \sum_{j=1}^{i} t_j\right)$, $w_i = \sum_{j=i+1-p}^{i+p} h_j,\ \ w = \sum_{j=1}^{m} w_i$, and $\tau_i = \frac{1}{m+1}\left(\sum_{j=i-p}^{m-p} t_j + \sum_{j=1+p}^{i+p} t_j\right)$.

With these notations, we can write succesively

$$\begin{aligned}
\theta_m - \theta_{m-r} &= \frac{1}{m}\sum_{i=1}^{m}(t_i - t_{i-r}) = \frac{1}{m}w',\\
\theta_{m+p} - \theta_{m-p} &= \frac{1}{m}\sum_{i=1}^{m}(t_{i+p} - t_{i-p}) = \frac{1}{m}w,\\
\mu_m^{(2)} - \mu_{m-r}^{(2)} &= \frac{2}{m\,(m+1)}\sum_{i=1}^{m} w'_i\tau'_i,\\
\mu_{m+p}^{(2)} - \mu_{m-p}^{(2)} &= \frac{2}{m\,(m+1)}\sum_{i=1}^{m} w_i\tau_i,
\end{aligned}$$

and the inequality $\gamma_{m,-r} \le \alpha_{m,-r}$ is equivalent to

$$\frac{1}{w'}\sum_{i=1}^{m} w'_i\tau'_i \le \frac{1}{w'}\sum_{i=1}^{m} w'_i\tau'_i.$$

It can be interpreted as follows: the barycenter of the m points τ'_i with weights w'_i is less than or equal to the barycenter of the m points τ_i with weights w_i. The inequality $\alpha_{m,-r} - \gamma_{m,-r} \le 1$, $r \in \{1, \ldots, p-1\}$ is equivalent to $W(-r,0,p) - W(-p,-r,0) \le W$ and can be written

$$\frac{\mu_{m-r}^{(2)} - \mu_{m-p}^{(2)}}{\theta_{m-r} - \theta_{m-p}} \le \frac{\mu_{m+p}^{(2)} - \mu_{m-r}^{(2)}}{\theta_{m+p} - \theta_{m-r}}.$$

Using the same techniques as above, we obtain successively

$$\begin{aligned}
\theta_{m-r} - \theta_{m-p} &= \frac{1}{m}\sum_{i=1}^{m}(t_{i-r} - t_{i-p}) = \frac{1}{m}\overline{w}',\\
\theta_{m+p} - \theta_{m-r} &= \frac{1}{m}\sum_{i=1}^{m}(t_{i+p} - t_{i-r}) = \frac{1}{m}\overline{w},
\end{aligned}$$

where

$$\overline{w}' = \sum_{i=1}^{m}\overline{w}'_i, \quad \text{and} \quad \overline{w} = \sum_{i=1}^{m}\overline{w}_i$$

with

$$\overline{w}'_i = \sum_{j=i+1-p}^{i+r} h_j, \quad \text{and} \quad \overline{w}_i = \sum_{j=i+1-r}^{i+p} h_j,$$

for $1 \leq i \leq m$. Defining $\overline{\tau}'_i = \frac{1}{m+1}\left(\sum_{j=i-p}^{m-p} t_j + \sum_{j=1-r}^{i-r} t_j\right)$, and $\overline{\tau}'_i = \frac{1}{m+1}\left(\sum_{j=i-r}^{m-r} t_j + \sum_{j=1+p}^{i+p} t_j\right)$ for $1 \leq i \leq m$, we get

$$\mu^{(2)}_{m-r} - \mu^{(2)}_{m-p} = \frac{2}{m\,(m+1)} \sum_{i=1}^{m} \overline{w}'_i \overline{\tau}'_i,$$

$$\mu^{(2)}_{m+p} - \mu^{(2)}_{m-r} = \frac{2}{m\,(m+1)} \sum_{i=1}^{m} \overline{w}_i \overline{\tau}_i.$$

Then the inequality $\alpha_{m,-r} - \gamma_{m,-r} \leq 1$ can be interpreted in a geometric form as follows: the barycenter of the m points $\overline{\tau}'_i$ with weights $\overline{w}'_i$ is less than or equal to the barycenter of the m points $\overline{\tau}_i$ with weights $\overline{w}_i$.
In a similar way, one can prove that inequalities $0 \leq \gamma_{m,s} - \alpha_{m,s} \leq 1$, for all $1 \leq s \leq p-1$ are equivalent to the following inequalities:

$$\frac{\mu^{(2)}_{m+p} - \mu^{(2)}_{m-p}}{\theta_{m+p} - \theta_{m-p}} \leq \frac{\mu^{(2)}_{m+s} - \mu^{(2)}_{m}}{\theta_{m+s} - \theta_{m}}, \quad \frac{\mu^{(2)}_{m+s} - \mu^{(2)}_{m-p}}{\theta_{m+s} - \theta_{m-p}} \leq \frac{\mu^{(2)}_{m+p} - \mu^{(2)}_{m+s}}{\theta_{m+p} - \theta_{m+s}},$$

and can also be interpreted in terms of barycenters of knots. Here we need the weighted points (τ''_i, w''_i) and $(\overline{\tau}''_i, \overline{w}''_i)$, where $\tau''_i = \frac{1}{m+1}\left(\sum_{j=i}^{m} t_j + \sum_{j=1+s}^{i+s} t_j\right)$, $\overline{\tau}''_i = \frac{1}{m+1}\left(\sum_{j=i+s}^{m+s} t_j + \sum_{j=1+p}^{i+p} t_j\right)$, $w''_i = \sum_{j=i+1+s}^{i+p} h_j$, and $\overline{w}''_i = \sum_{j=i+1}^{i+s} h_j$.

Theorem 9 *Assume that the sequence of knots T satisfies, for all $i \in \mathbb{Z}$, the following properties:*

1. *for all $1 \leq r \leq p-1$, the barycenter of the m points (τ'_i, w'_i) (resp. $(\overline{\tau}'_i, \overline{w}'_i)$) is less than or equal to the barycenter of the m points (τ_i, w_i) (resp. $(\overline{\tau}_i, \overline{w}_i)$).*

2. *for all $1 \leq s \leq p-1$, the barycenter of the m points (τ_i, w_i) (resp. $(\overline{\tau}_i, \overline{w}_i)$) is less than or equal to the barycenter of the m points (τ''_i, w''_i) (resp. $(\overline{\tau}''_i, \overline{w}''_i)$).*

Then, for all $i \in \mathbb{Z}$, a^*_i is an optimal solution of the local minimization problem $(\bar{M}_i)$. Thus, for all $p \geq m$, the spline iQIs G^*_p of theorem 9 are near-best.

Remark 3 *Even if the partition T does not satisfy the hypothesis of theorem 9, the operator G^*_p is still a good iQI because its infinity norm is uniformly bounded (theorem 8).*

11 Quasi-Interpolants exact on $\mathbb{P}_n$ with $n \geq 3$

While we succeeded above in characterizing some families of near-best QIs exact on $\mathbb{P}_2$ whose norms are *uniformly bounded independently of the partition*, it is

surprisingly difficult to find QIs exact on $\mathbb{P}_3$ having the same property. Actually, we did not find any example of such a QI.
For example, let us consider the following cubic spline dQI (which also appears in [20]), defined on $I = \mathbb{R}$ endowed with a non-uniform partition, by

$$Q_3 f(x) = \sum_{i\in\mathbb{Z}} \lambda_i(f) B_i(x),$$

where B_i is the cubic B-spline with support $[t_{i-2}, t_{i+2}]$ centered at t_i and

$$\lambda_i(f) = a_i f_{i-1} + b_i f_i + c_i f_{i+1},$$

where $f_i = f(t_i)$, and the coefficients are given by

$$a_i = -\frac{1}{3}\frac{h_i^2}{h_{i-1}(h_{i-1}+h_i)}, \;\; b_i = \frac{1}{3}\frac{(h_{i-1}+h_i)^2}{h_{i-1}h_i}, \;\; c_i = -\frac{1}{3}\frac{h_{i-1}^2}{h_i(h_{i-1}+h_i)}.$$

It is easy to verify that Q_3 is *exact on* $\mathbb{P}_3$, i.e. that a_i, b_i, c_i satisfy the system of linear equations:

$$a_i + b_i + c_i = 1, \quad t_{i-1}a_i + t_i b_i + t_{i+1}c_i = \theta_i,$$

$$t_{i-1}^2 a_i + t_i^2 b_i + t_{i+1}^2 c_i = \theta_i^{(2)}, \quad t_{i-1}^3 a_i + t_i^3 b_i + t_{i+1}^3 c_i = \theta_i^{(3)}.$$

This operator can also be written in the quasi-Lagrange form

$$Q_3 f(x) = \sum_{i\in\mathbb{Z}} f_i \tilde{B}_i(x),$$

where the fundamental function $\tilde{B}_i$, having support $[t_{i-3}, t_{i+3}]$, is defined by

$$\tilde{B}_i = c_{i-1}B_{i-1} + b_i B_i + a_{i+1}B_{i+1}.$$

The Chebyshev norm $|\Lambda|_\infty$ of the associated Lebesgue function $\Lambda = \sum_{i\in\mathbb{Z}} |\tilde{B}_i|$ satisfies:

$$|\Lambda|_\infty = \|Q_3\|_\infty.$$

For $x \in I = [t_2, t_3]$, we have $\Lambda = \sum_{i=0}^{5} |\tilde{B}_i|$ and we shall now construct a *partition for which* $|\Lambda|_\infty$ *is arbitrary large*. We choose $h_3 = h$ as parameter and $h_i = 1$ for all $i \neq 3$. Then the fundamental functions on the interval I are given by :

$$\tilde{B}_0 = -\frac{1}{6}B_1, \;\; \tilde{B}_1 = \frac{4}{3}B_1 - \frac{1}{6}B_2, \;\; \tilde{B}_2 = -\frac{1}{6}B_1 + \frac{4}{3}B_2 - \frac{h^2}{3(1+h)}B_3,$$

$$\tilde{B}_3 = -\frac{1}{6}B_2 + \frac{(1+h)^2}{3h}B_3 - \frac{1}{3}\frac{1}{h(1+h)}B_4,$$

$$\tilde{B}_4 = -\frac{1}{3h(1+h)}B_3 + \frac{1}{3}\frac{(h+1)^2}{h}B_4, \;\; \tilde{B}_5 = -\frac{1}{3}\frac{h^2}{(1+h)}B_4.$$

Let us compute the value of Λ at the midpoint $s = \frac{1}{2}(t_2 + t_3)$ of I.
Setting $\alpha_1 = B_1(t_1) = B_4(t_3)$, $\alpha_2 = B_2(t_1) = B_3(t_3)$, $\delta_2 = B_2(t_3) = B_3(t_2)$ and using the algebraic properties of B-splines, we get

$$\alpha_1 = \frac{h}{3(1+h)}, \quad \alpha_2 = \frac{2h^3 + h^2 + 9}{3(h+1)(h^2 - h + 3)}, \quad \delta_2 = \frac{h}{(h+1)(h^2 - h + 3)}.$$

(we have $\alpha_1 + \alpha_2 + \delta_2 = 1$ because the B-splines sum to one). Now, using the values of the central BB-coefficients β_2 and γ_2 of B_2 on the interval I:

$$\gamma_2 = \frac{1}{h^2 - h + 3}, \quad \beta_2 = 1 - \gamma_2,$$

we can compute the values of B-splines at this point

$$B_1(s) = \frac{1}{8}\alpha_1 = B_4(s), \quad B_2(s) = B_3(s) = \frac{1}{8}(\alpha_2 + 3\beta_2 + 3\gamma_2 + \delta_2) = \frac{1}{8}(4 - \alpha_1).$$

After some algebraic calculations, we obtain the asymptotic behaviour of the Lebesgue function at the midpoint of I :

$$\Lambda(s) = O(h), \quad h \to +\infty,$$

therefore the norm of the associated QI is unbounded.

However, if we now assume that there exists $r > 0$ such that the partition satisfies

$$\frac{1}{r} \leq \frac{h_{i+1}}{h_i} \leq r, \quad i \in \mathbb{Z},$$

then we obtain the following upper bounds:

$$|a_i|, |c_i| \leq \frac{1}{3}\frac{r^2}{(1+r)}, \quad |b_i| \leq \frac{1}{3}(1+r)^2,$$

from which we deduce

$$\|Q_3\|_\infty \leq N(r) := \frac{1}{3}\left((1+r)^2 + \frac{2r^2}{(1+r)}\right).$$

For example, for $r = 1, 2, 3, 4, 5$, we get the following values of the upper bound of the norm :

$$N(1) \approx 1.66, \quad N(2) \approx 3.89, \quad N(3) \approx 6.83, \quad N(4) \approx 10.47, \quad N(5) \approx 14.78,$$

which are still of reasonable size. Therefore it seems that such QIs are interesting in practice though they are not uniformly bounded with respect to *all* partitions.

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